Some benchmark solutions for the tsunami wave rays and fronts

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Introduction

Methods for tsunami kinematics are the effective instrument that is still widely used for estimating the tsunami travel times They are based on the Lagrange formula for the tsunami propagation velocity

$$c = \sqrt{gD} \quad ,$$

where g is the acceleration of the gravity and D is the depth.

Some definitions:

Wave front is the interface between undisturbed and disturbed water areas.

Wave rays are the lines which are always orthogonal to the wave front. On the other hand, the wave ray presents the optimal (quickest) route for wave propagation.

There are some governing equations for the **tsunami front** kinematics.

The Eikonal Equation

$$(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = \frac{1}{v^2(x,y)} \quad ,$$

where $v(x,y)$ gives the velocity distribution in a medium, function $f(x,y)=T$ presents the wave front line at the time instance T .

And for the **wave ray** trajectory

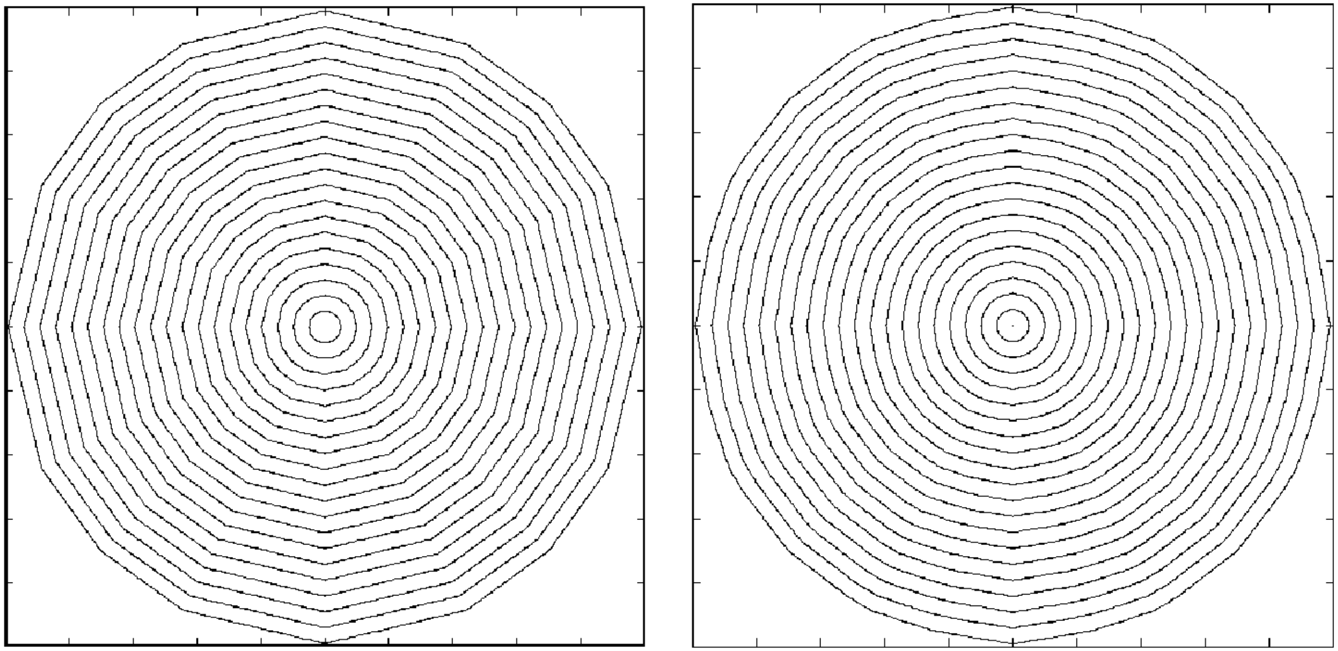
The differential equations of a wave ray

$$\frac{d\vec{x}}{dt} = v^2(\vec{x}) \times \vec{p} \quad , \qquad \frac{d\vec{p}}{dt} = \nabla \ln \frac{1}{v} \quad ,$$

$\vec{p}$ is the ray direction vector

For some models of bottom topography it is possible to obtain the exact analytical formulas for the ray trajectory.

For the trivial case of ***constant depth*** the tsunami wave front from the circled or the point source, obviously will be a circle of increasing radius. Wave rays are presented by its radial straight lines. Tsunami travel time between any two points of this area is expressed as the distance divided by the constant propagation velocity. This benchmark solution was used for testing the method for travel-time computation based on Huygens principle [Marchuk, 2008] .



[1]. Marchuk An.G. Minimizing computational errors of tsunami wave-ray and travel-time // Science of tsunami hazards, Vol. 27, No. 4, 2008, pp. 12-24.

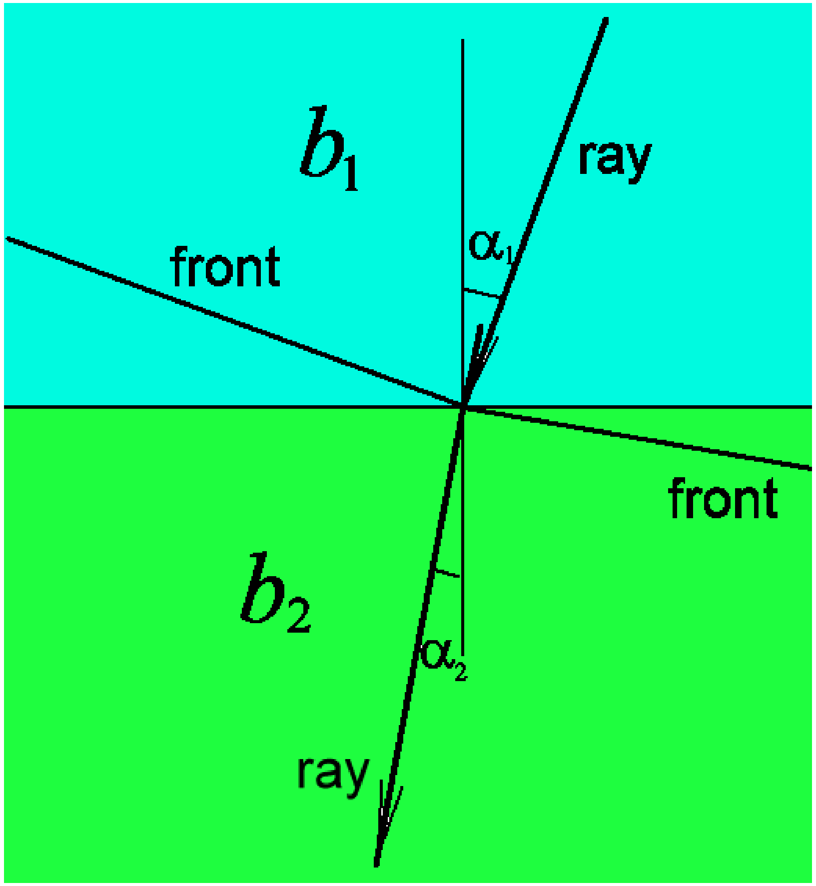
The sloping bottom topography

Let the depth linearly increases proportionally a distance to the straight shoreline

$D(x,y)=ay$. Tsunami propagation velocity: $v(x,y)=\sqrt{a\times g\times y}=a_1\sqrt{y}$.

The exact solution for the wave ray trajectory above te bottom slope has been derived in [2] using the wave (ray) refraction law on the interface between two media having different conductivity (wave propagation velocities) b_1 and b_2

$$\frac{\sin(\alpha_1)}{b_1}=\frac{\sin(\alpha_2)}{b_2}\quad .$$



[2]. Marchuk An.G. Benchmark solutions for tsunami wave fronts and rays. Part 1: sloping bottom topography. Science of Tsunami Hazards, Vol. 35, N2, 2016, pp. 34-48.

As it was shown in [2], above the bottom slope the wave ray is presented by the arc of cycloid which can be expressed in parametric form as

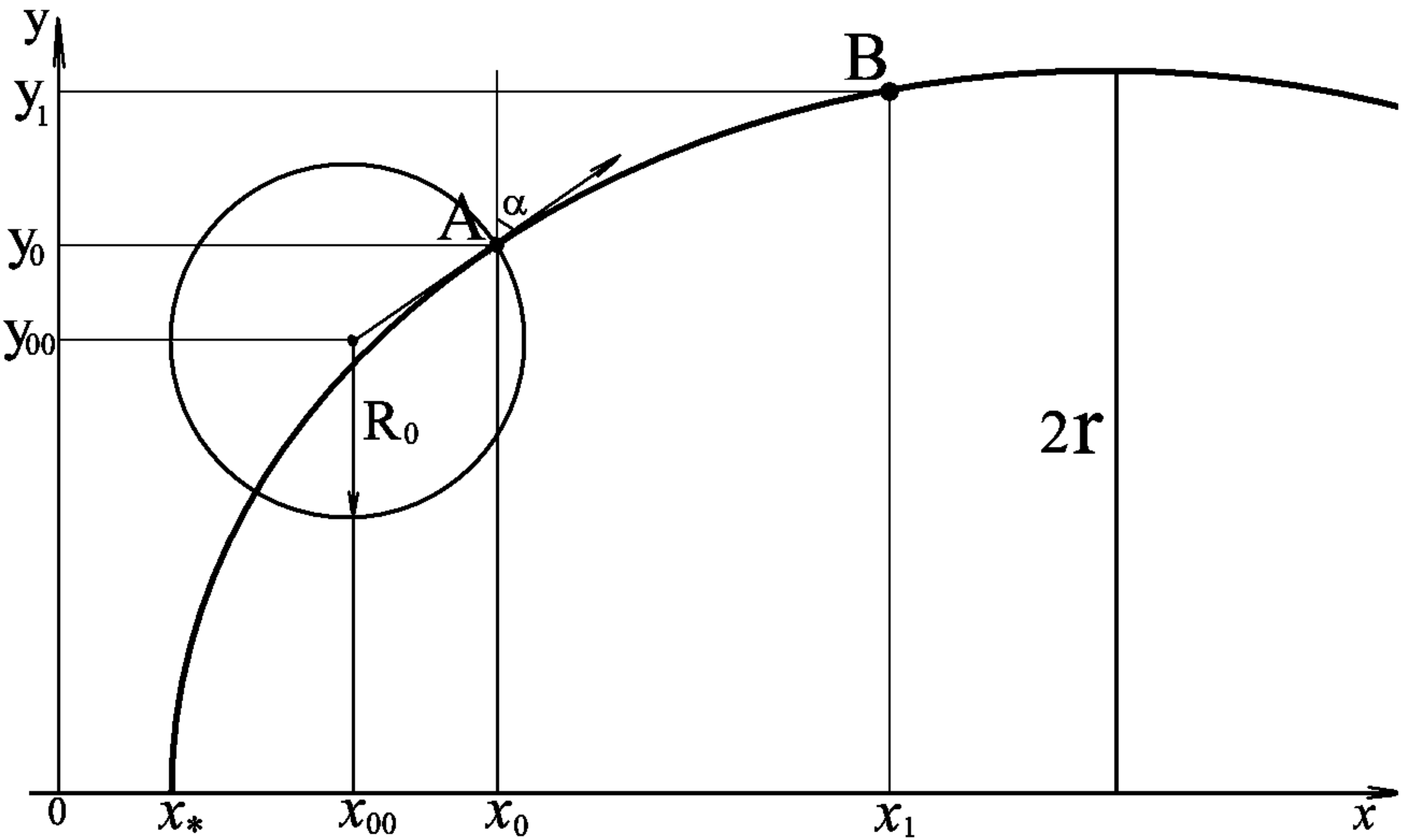
$$x = r(u - \sin(u)) + x_* \ ,$$

$$y = r(1 - \cos(u)), \qquad u \in [0, 2\pi] \ .$$

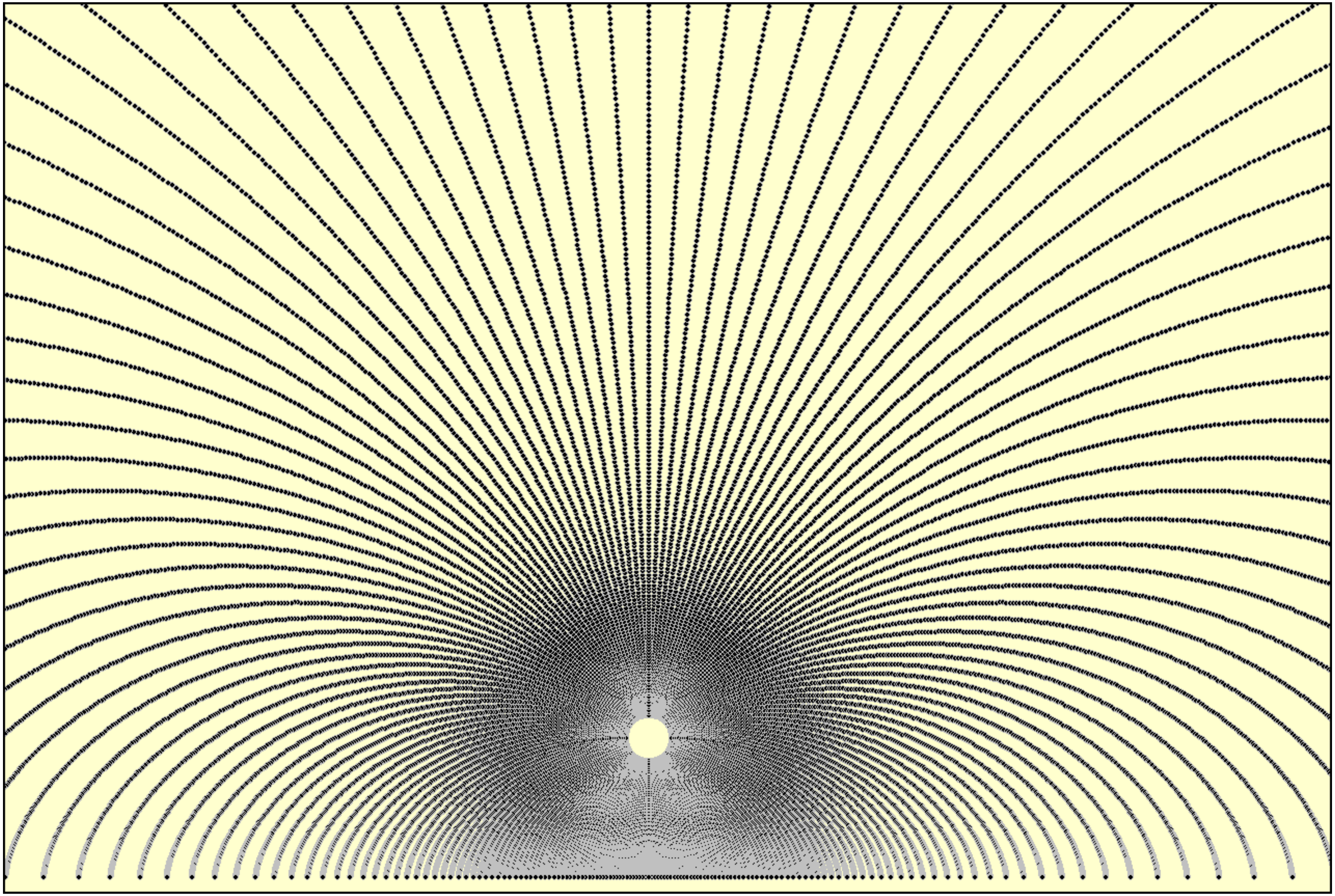
Here r is the cycloid radius, u is the parameter which meaning is the doubled angle between the ray direction and the ordinate axis. x_* is the coordinate of the cycloid exit point on the X-axis.

If at the distance y_1 off the coastline the angle between ray direction and the orthogonal to the coastline is equal to α_1, then

$$r = \frac{y_1}{2 \cdot \sin(\alpha_1)} \ .$$



The scheme of determination the wave ray exiting the boundary of the circle-shaped source



. Wave-ray traces above the sloping bottom topography coming from 200 points locating at the circled source boundary

The tsunami travel time between points **A** and **B** along wave ray is equal to

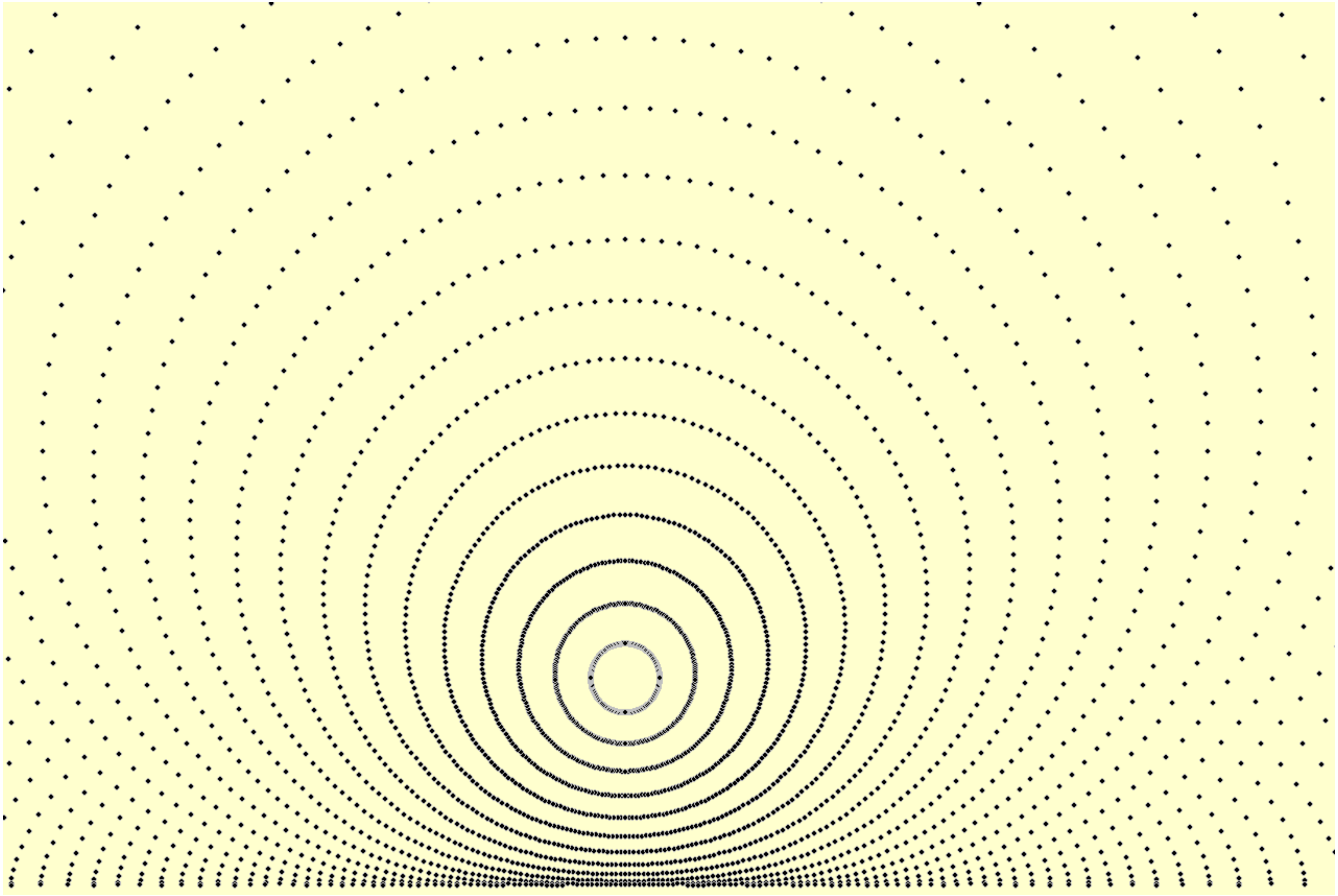
$$T = \int_A^B \frac{ds}{\sqrt{g \cdot a \cdot y}} = (u_B - u_A)\sqrt{\frac{2r}{g \cdot a}}$$, were $u_A = 2\alpha_1$ and $u_B = \arccos\left(1 - \frac{y_2}{r}\right)$

If we express coordinates of the point **B** as a function of T and α_1 then it is possible to build tsunami isochrones

$$x_1 = \frac{y_0}{1-\cos(2\alpha)}\left(\left(2\alpha + T\sqrt{\frac{g \cdot a \cdot (1-\cos(\alpha))}{2y_0}} - \sin\left(2\alpha + T\sqrt{\frac{g \cdot a \cdot (1-\cos(\alpha))}{2y_0}}\right)\right)\right) + x_*$$

$$y_1 = \frac{y_0}{1-\cos(2\alpha)}\left(1 - \cos\left(2\alpha + T\sqrt{\frac{g \cdot a \cdot (1-\cos(\alpha))}{2y_0}}\right)\right)$$

$0 > \alpha > \pi/2$



Positions of the tsunami wave front (isochrones) within the 5-minutes interval from a circled source of radius 50 km above the sloping bottom topography

The parabolic bottom topography

In the case when a depth increases proportionally the squared offshore distance

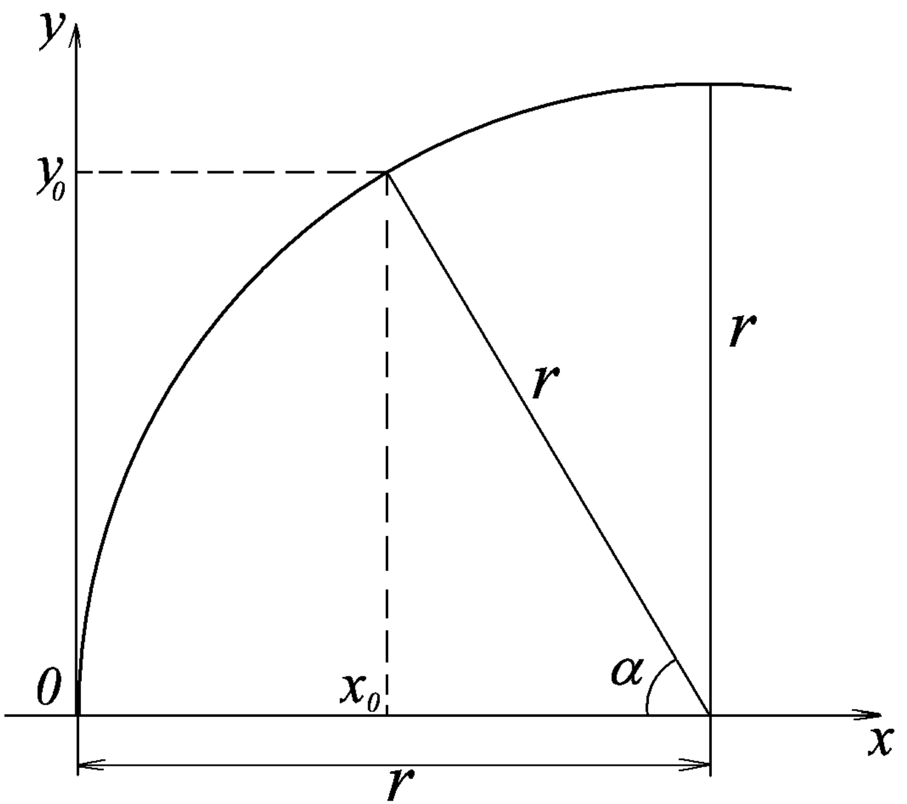
$$D(x,y)=ky^2$$

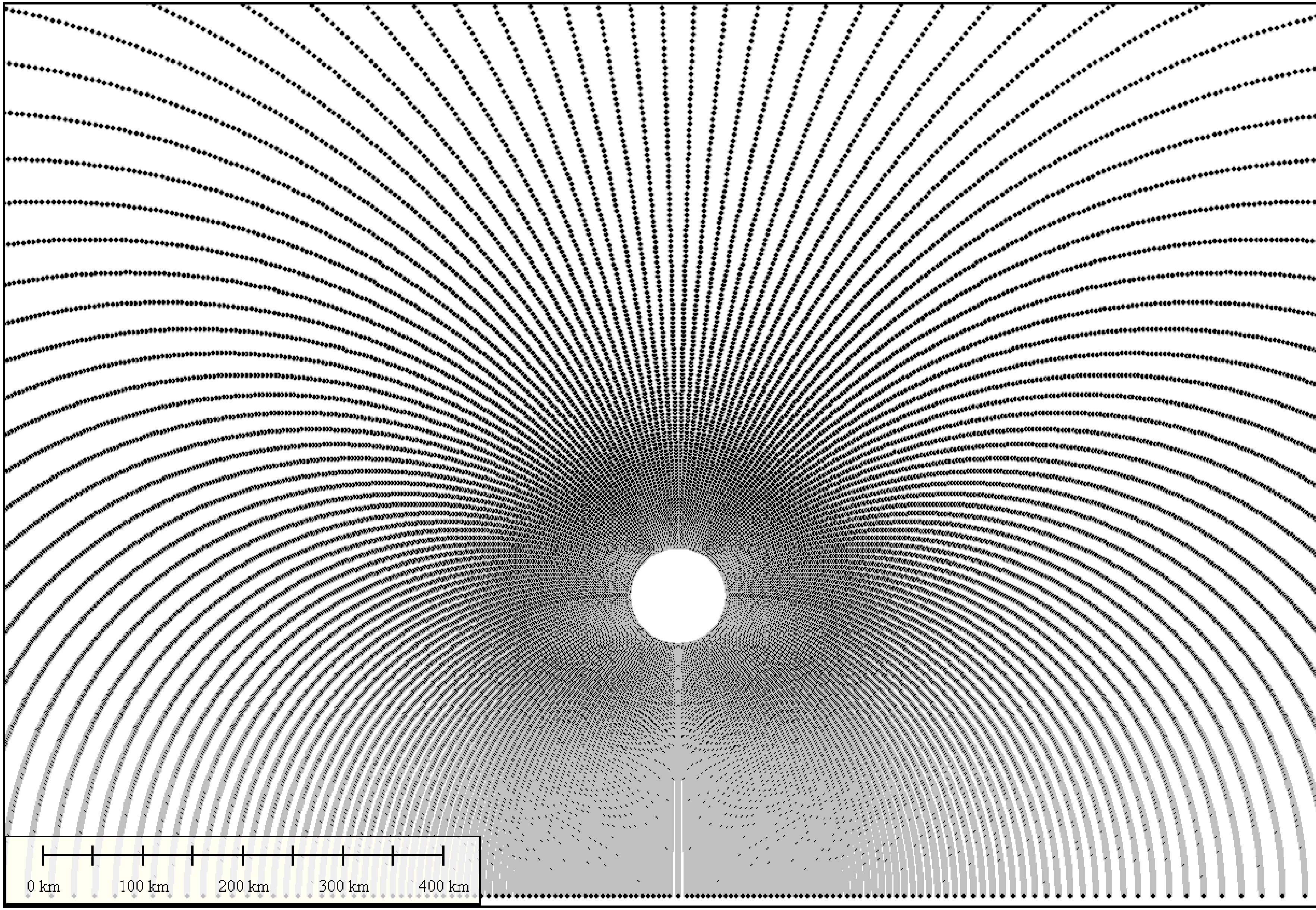
the wave-ray trace is presented by the arc of a circle having center at the coastline (y=0)

$$(r-x)^2+y^2=r^2$$

$$r=\frac{x_0^2+y_0^2}{2x_0}$$

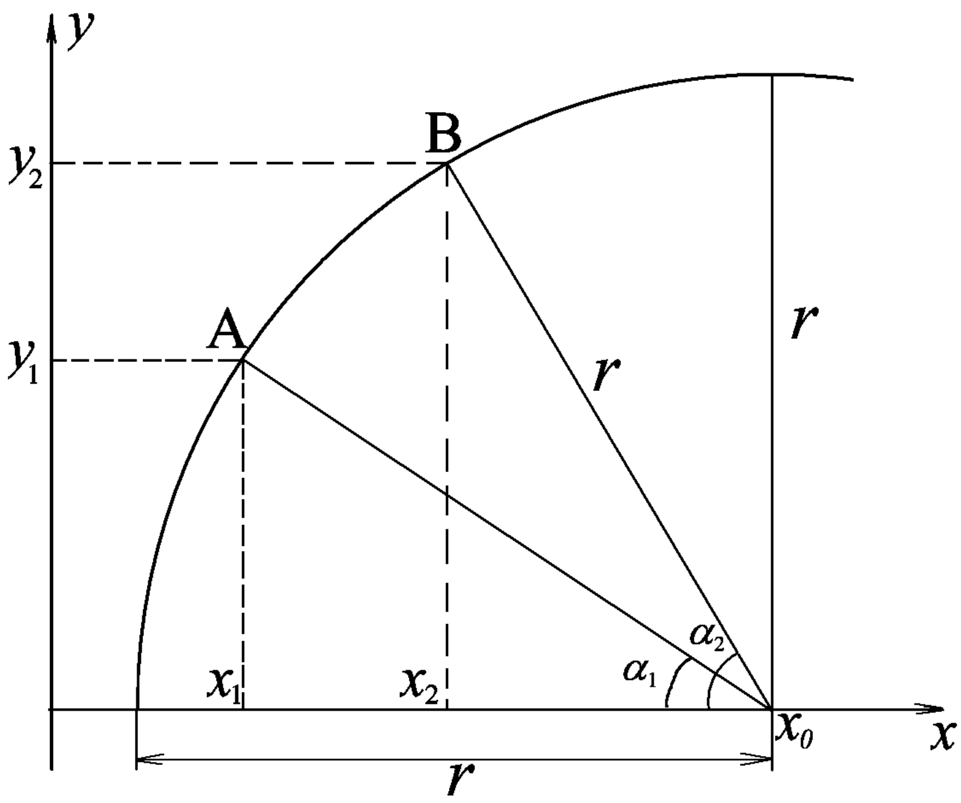
$$y=\sqrt{x(2r-x)} \quad , \quad x\in[0,2r]$$





Wave-ray traces above the parabolic bottom topography coming from 200 points locating at the circled source boundary

Tsunami **travel time** from point **A** to point **B** along the wave ray (arc of a circle) is equal to



$$T = \frac{1}{\sqrt{kg}} \int_{\alpha_1}^{\alpha_2} \frac{d\alpha \cdot r}{r \cdot \sin(\alpha)} = \frac{1}{k_1} \ln\left| tg\left(\frac{\alpha}{2}\right)\right|\Bigg\|_{\alpha_1}^{\alpha_2} =$$

$$= \frac{1}{k_1}\left(\ln\left| tg\left(\frac{\alpha_2}{2}\right)\right| - \ln\left| tg\left(\frac{\alpha_1}{2}\right)\right|\right)$$

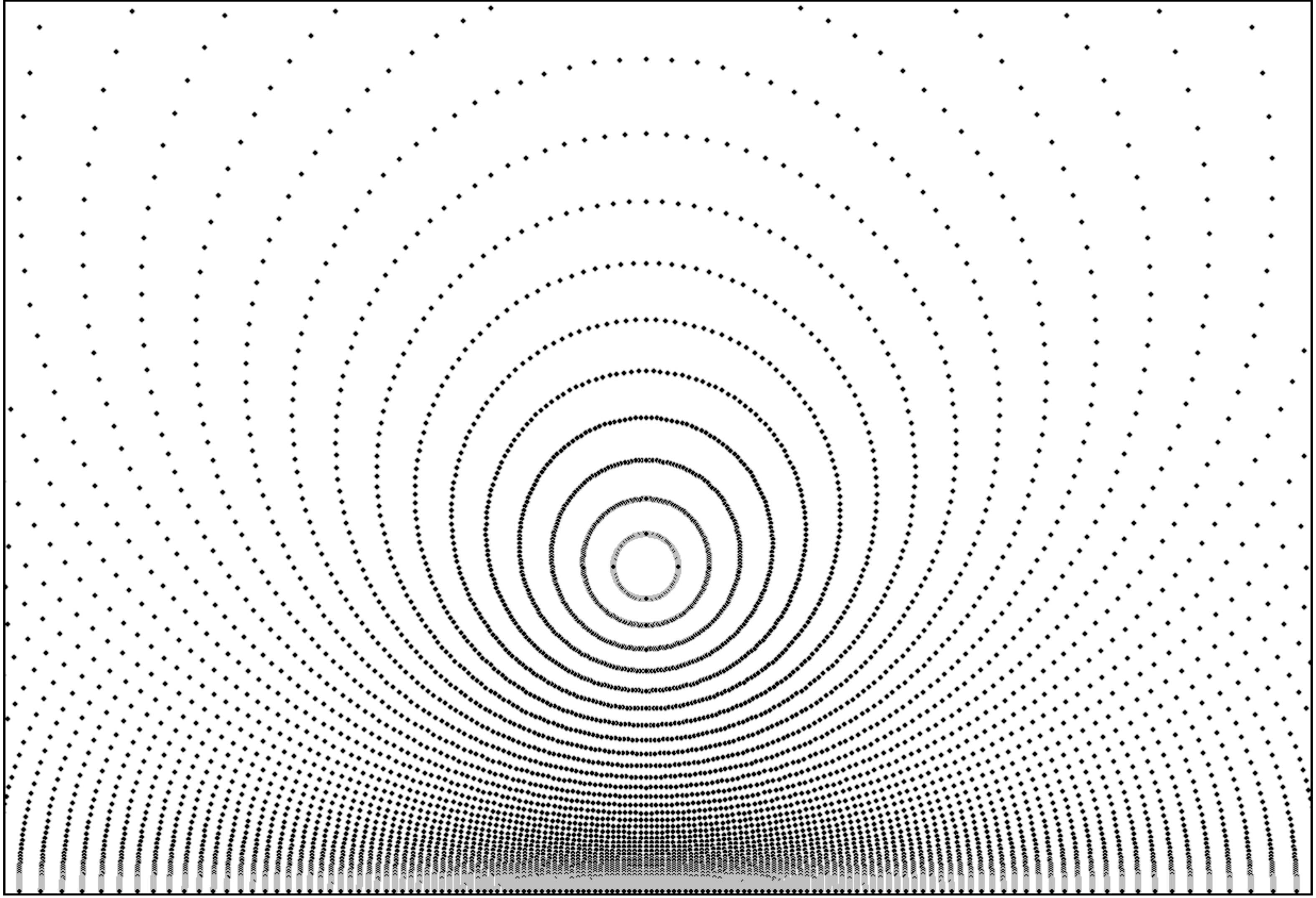
where $k_1 = \sqrt{kg} = const$

It is possible to express the angle α_2 through the angle α_1 and the time T

$$\alpha_2 = 2arctg\left(\exp(k_1 T) \cdot tg\left(\frac{\alpha_1}{2}\right)\right), \quad 0 \leq \alpha_1 < \alpha_2 < \pi$$

Finally, the coordinates of the destination point which the wave front will reach at the time T going along the wave ray, can be written down as

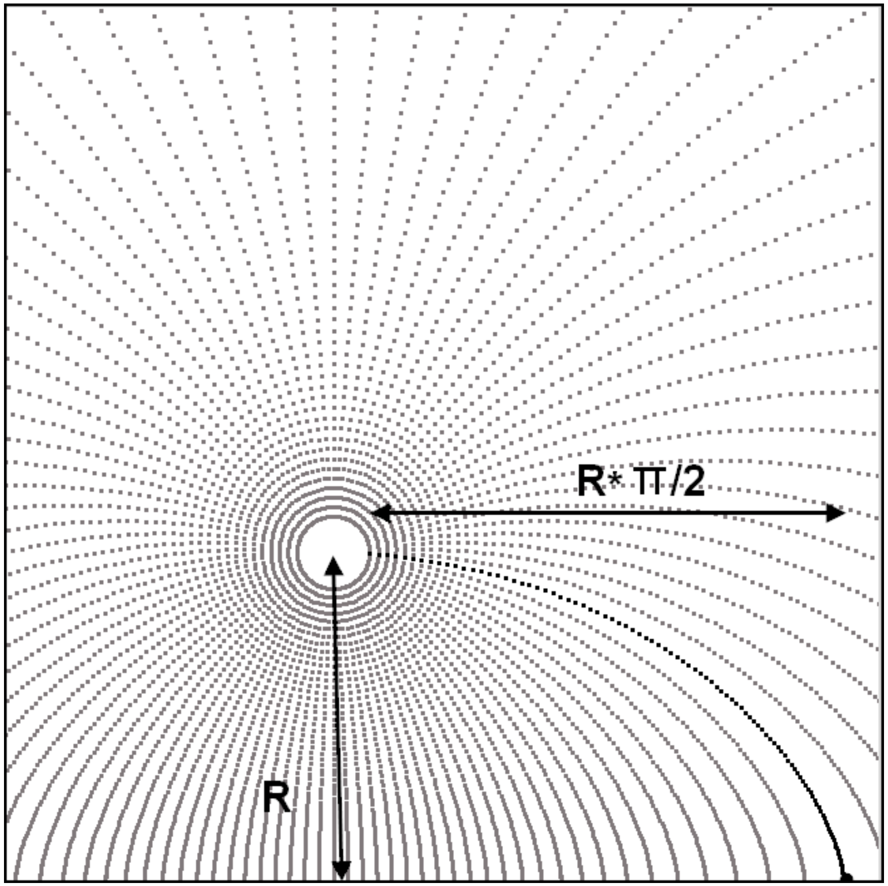
$$x_2(T,\alpha_1) = x_0 - r \cdot \cos(\alpha_2) \qquad y_2(T,\alpha_1) = r \cdot \sin(\alpha_2)$$



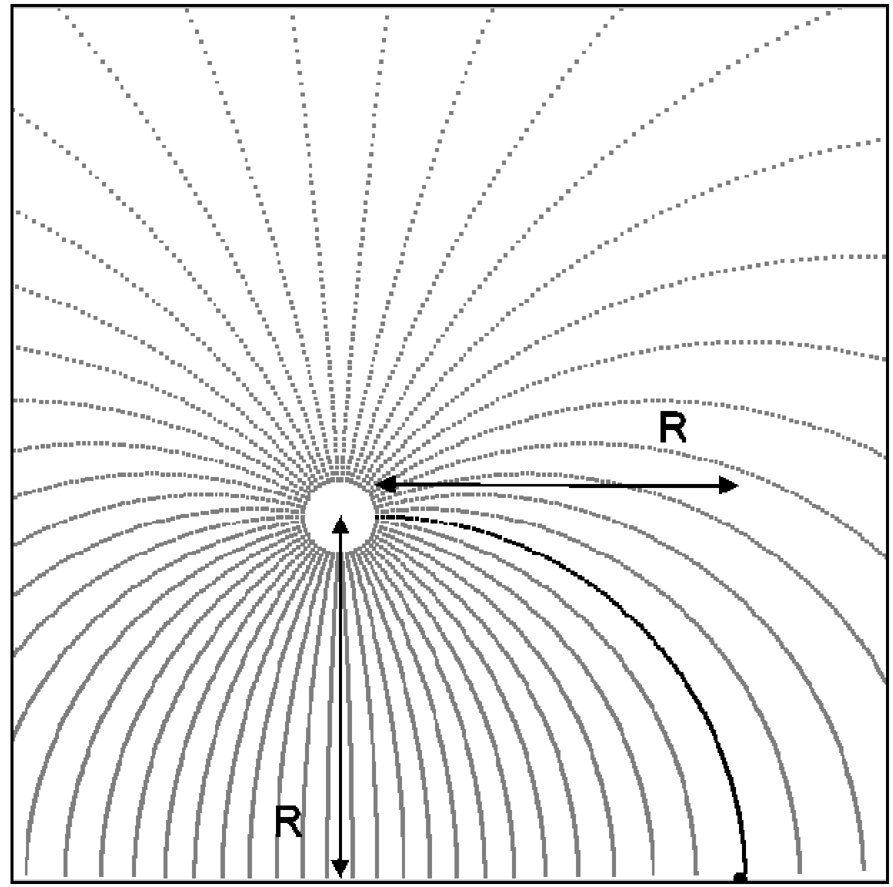
Positions of the tsunami wave front (isochrones) within the 5-minutes interval from a circled source of radius 50 km above the parabolic bottom topography. The wave front at any time instance is presented by a circle [3].

[3]. Borovskikh A.V. Two-dimensional eikonal equation // Siberian Mathematical Journal – 2006.-Vol. 47, N5 - pp.993-1018 (in Russian).

Test for the wave ray numerical computation

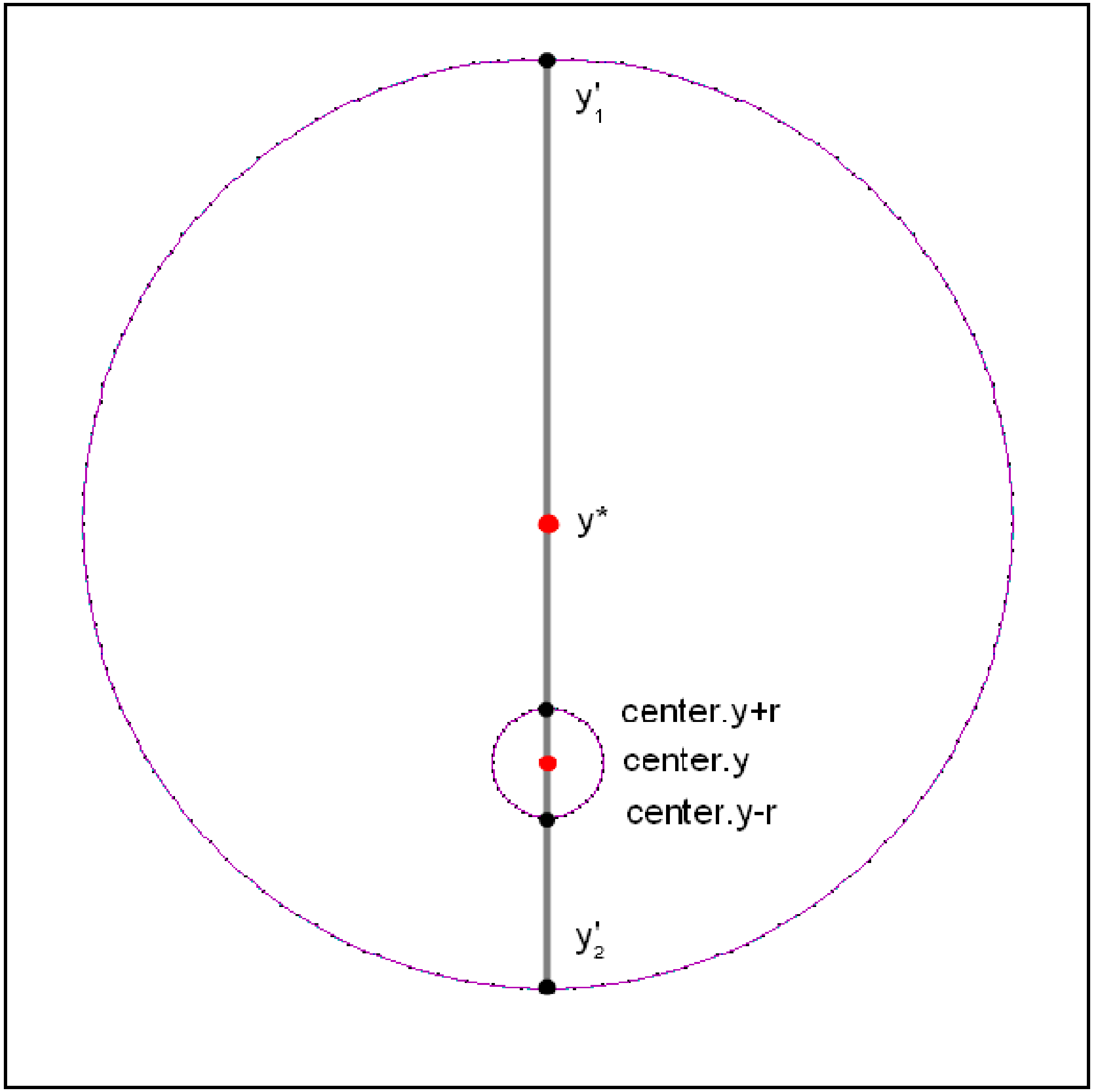


The sloping bottom topography



The parabolic bottom topography

Test for the wave front kinematics computation

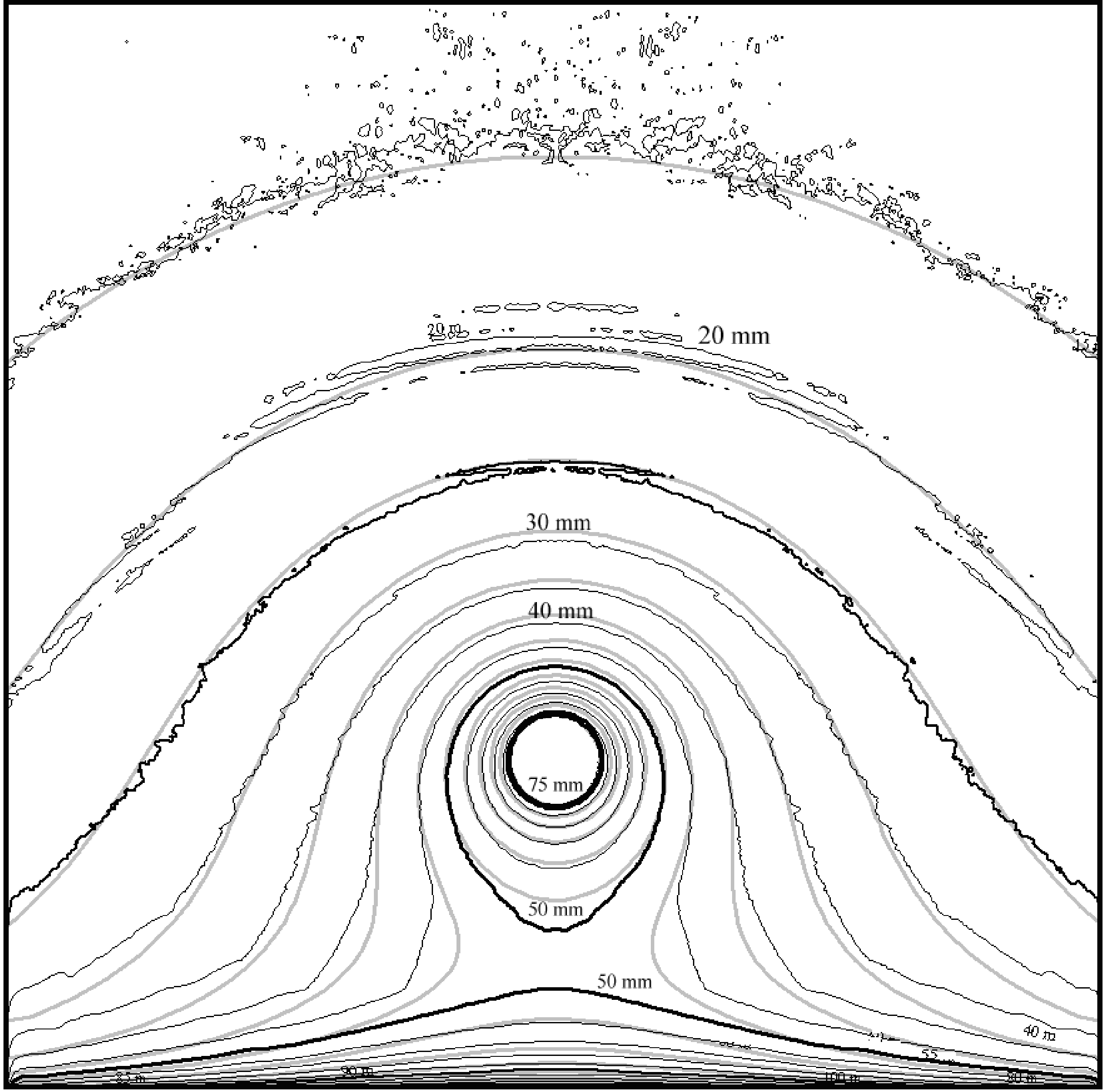


Comparison of the calculated and exact wave front above the parabolic bottom

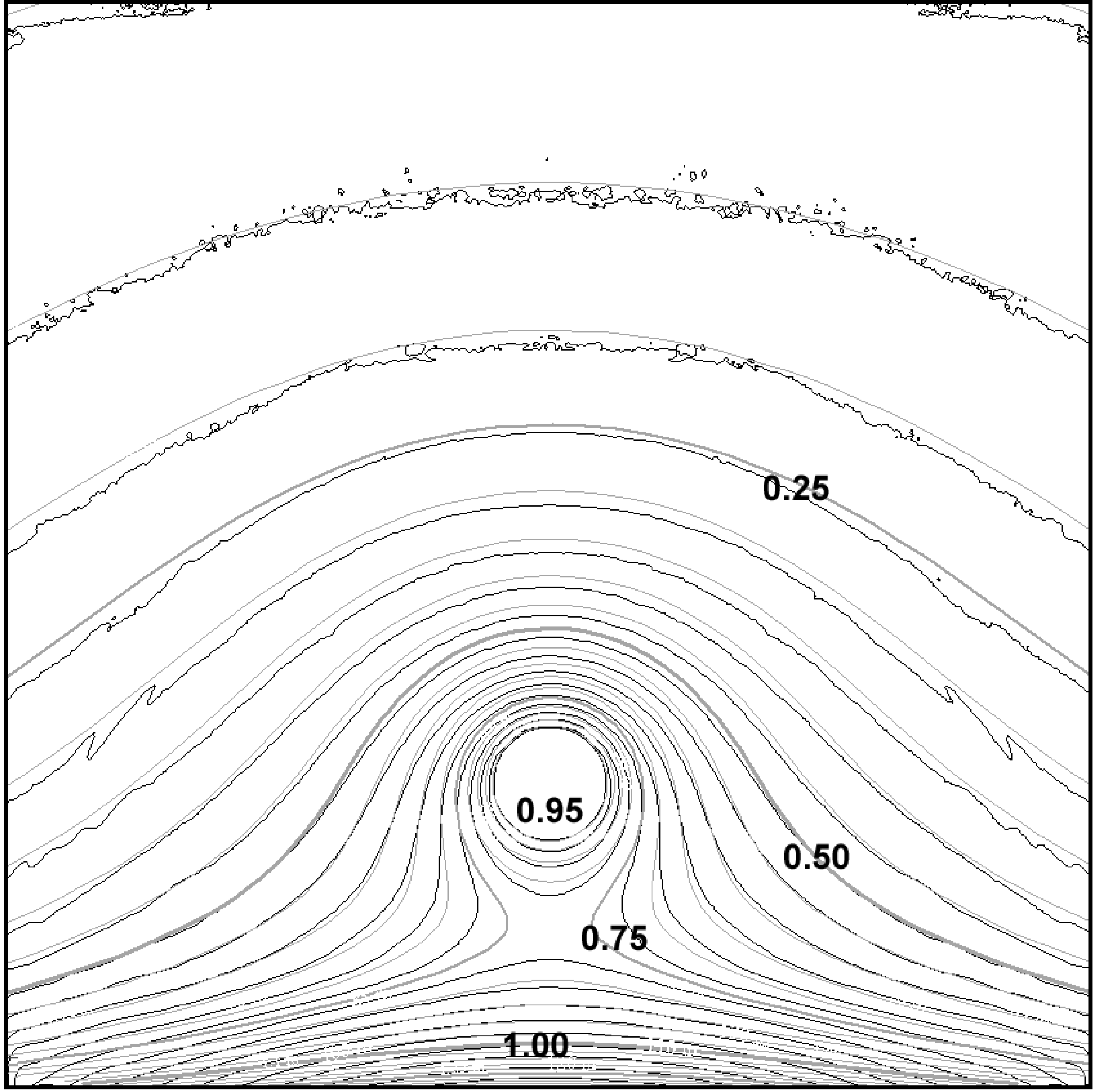
Conclusion

The following benchmark solutions for tsunami wave rays and fronts were obtained:

1. In the area with the sloping bottom topography all the wave rays are presented by arcs of cycloid
2. Formulas for calculation coordinates of the points located along tsunami isochrones also have been derived for the case of sloping bottom.
3. When a depth increases proportionally the squared distance to the straight shoreline, the wave ray trajectory is presented by the arc of circle having center at a coastline.
4. Formulas for determination the tsunami isochrones also have been derived for the case of parabolic bottom topography.
5. All solutions obtained make it possible to estimate tsunami wave heights using the ray approximation.



Comparative location of isolines of tsunami height maxima calculated by the numerical shallow water model (Titov, Gonzalez, 1997) (black color) and by the ray approximation (grey color). The sloping bottom topography



Comparative location of isolines of tsunami height maxima calculated by the numerical shallow water model (black color) and by the ray approximation (grey color). The parabolic bottom topography

Thank you!